Game-theoretic versions of strong law of large numbers for unbounded variables

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Abstract

We consider strong law of large numbers (SLLN) in the framework of game-theoretic probability of Shafer and Vovk (2001). We prove several versions of SLLN for the case that Reality's moves are unbounded. Our game-theoretic versions of SLLN largely correspond to standard measure-theoretic results. However game-theoretic proofs are different from measure-theoretic ones in the explicit consideration of various hedges. In measure-theoretic proofs existence of moments are assumed, whereas in our game-theoretic proofs we assume availability of various hedges to Skeptic for finite prices.

Keywords and phrases: Borel-Cantelli lemma, call option, Doob's upcrossing lemma, Kronecker's lemma, Marcinkiewicz-Zygmund strong law, martingale convergence theorem.

1 Introduction

In the framework of game-theoretic probability, proof of SLLN is simple if Reality's moves are bounded. In [4] we showed that a single simple strategy based on past averages of Reality's moves forces SLLN for the case of bounded Reality's moves. For the special case of the coin-tossing game path behavior and convergence rate of SLLN can be very explicitly

stated ([5],[8]). However when Reality's moves are not bounded, the proof becomes more complicated due to consideration of availability of hedges to Skeptic. Under the requirement of the collateral duty that Skeptic has to keep his capital always nonnegative, he has to use some form of hedge at each round. In Chapter 4 of Shafer and Vovk (2001), Kolmogorov's SLLN is proved under the availability of the variance hedge (quadratic hedge). Shafer and Vovk consider the case that the price of the variance hedge is announced by Forecaster for each round, but for simplicity in this paper we omit Forecaster from the protocol and consider the case that hedges carry constant prices throughout the game. Availability of the quadratic hedge is natural and convenient. However the purpose of this paper is to investigate SLLN under other types of hedges.

In measure-theoretic probability, the usual and most elegant form of SLLN is stated for the sample average $\bar{x}_n = (1/n)(x_1 + \cdots + x_n)$ of i.i.d. random variables, where only the existence of the measure-theoretic expected value $E|x_n| < \infty$ is assumed. However Kolmogorov's SLLN proved in Chapter 4 of Shafer and Vovk (2001) does not correspond to this version and a question remains whether a corresponding game-theoretic result holds or not. Some considerations of this problem are given in Chapter 4 of [9]. The usual measure-theoretic result depends strongly on the assumption of identical distribution of the random variables. On the other hand the basic feature of the game-theoretic probability is that the game is a martingale and there is a question of how to impose identical behavior to Reality at each round. In this paper we argue that the assumption of the identical distribution in measure-theoretic framework can be replaced by the availability of countable number of weak hedges.

For the most part we follow the standard proofs of SLLN in measure-theoretic probability. For example we use truncation and Kronecker's lemma. However our proofs differ from standard measure-theoretic proofs in explicit construction of Skeptic's strategy which requires Skeptic to observe his collateral duty. In addition our proof is more an extension of the proof for the bounded case of Chapter 3 of Shafer and Vovk (2001), rather than an extension of their proof in Chapter 4 using the quadratic hedge.

The organization of this paper is as follows. In Section 2 we set up notations and give some preliminary results. In Section 3 we prove a version of SLLN under the assumption of availability of a single hedge. In Section 4 we prove a game-theoretic version of SLLN for i.i.d. variables under the assumption of availability of countable hedges. We extend it to a Marcinkiewicz-Zygmund strong law in Section 5. Finally in Section 6 we discuss various aspects of our proofs and the assumption of availability of infinite number of hedges.

2 Notation and preliminaries

In this section we summarize our notations and some preliminary results. We follow the notation of Shafer and Vovk (2001). $\xi = x_1 x_2 \ldots$ denotes an infinite path of Reality's moves and $\xi^n = x_1 \ldots x_n$ denotes the partial path up to round n. For a strategy $\mathcal{P}$ of Skeptic, $\mathcal{K}_n^{\mathcal{P}}(\xi) = \mathcal{K}_n^{\mathcal{P}}(\xi^n)$ denotes the capital process. Starting with a positive initial

capital of $\mathcal{K}_0 = \delta > 0$, Skeptic observes his collateral duty by using $\mathcal{P}$ if

$$\mathcal{K}_n^{\mathcal{P}}(\xi) \geq 0, \quad \forall \xi, \forall n. \tag{1}$$

We also say that $\mathcal{P}$ satisfies the collateral duty with the initial capital δ. Note that $\mathcal{P}$ satisfies the collateral duty with initial capital δ if and only if $\mathcal{P}/\delta$ satisfies the duty with the initial capital 1. In view of this fact, we simply say that $\mathcal{P}$ satisfies the collateral duty if $\mathcal{P}$ satisfies the duty with some initial capital $\delta > 0$. When $\mathcal{P}$ satisfies the collateral duty, the capital process $\mathcal{K}^{\mathcal{P}}$ is called a (game-theoretic) non-negative martingale.

We call a function $h(x)$ of Reality's move x a *hedge* if it is non-negative ($h(x) \geq 0, \forall x \in \mathbb{R}$) and has a finite price $0 < \nu < \infty$. Skeptic is allowed to buy arbitrary amount of $h(x)$ with the unit price ν. In Chapter 4 of Shafer and Vovk (2001), they consider the variance hedge $h(x) = x^2$. In view of the unbounded forecasting game in Chapter 4 of Shafer and Vovk (2001), we first consider the following protocol with a single hedge.

The Unbounded Forecasting Game with a Single Hedge
Protocol:

$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \dots$:
  Skeptic announces $M_n \in \mathbb{R}$, $V_n \geq 0$.
  Reality announces $x_n \in \mathbb{R}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n + V_n(h(x_n) - \nu)$
END FOR

Availability of the variance hedge $h(x) = x^2$ is very convenient, because Skeptic can then construct a martingale which is a quadratic form of Reality's moves. This fact is used by Shafer and Vovk in their proof. However SLLN can be proved under other hedges. In Section 3 we will prove that SLLN is forced if the absolute moment hedge of order $1 + \epsilon$, $\epsilon > 0$,

$$h(x) = |x|^{1+\epsilon}$$

is available to Skeptic. Naturally we are tempted to consider the absolute moment hedge

$$h(x) = |x|$$

in the above protocol, corresponding to the measure-theoretic SLLN of i.i.d. random variables with finite expectation. However it is essential to point out that SLLN is not forced under the availability of $h(x) = |x|$ alone. Since this fact is important, we state it as a proposition. The following proposition is stated in view of the Marcinkiewicz-Zygmund strong law in Section 5.

Proposition 2.1. *Consider the unbounded forecasting game with a single hedge $h(x) = |x|^r$, $r > 0$. There exists no strategy $\mathcal{P}$ of Skeptic satisfying the collateral duty, such that $\lim_n \mathcal{K}_n^{\mathcal{P}} = \infty$ whenever $(x_1 + \cdots + x_n)/n^{1/r} \not\to 0$.*

Proof of this proposition, following Section 4.3 of Shafer and Vovk (2001), is given in Appendix A. Unfortunately it requires a measure-theoretic argument.

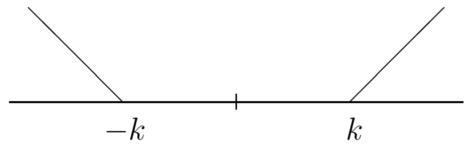

Figure 1: Symmetric call option type hedge

Because of Proposition 2.1 with $r = 1$, we need to assume that more hedges in addition to $h(x) = |x|$ are available to Skeptic in order to prove SLLN corresponding to the sample average of i.i.d. random variables with finite measure-theoretic expected value $E|x_n| < \infty$. Let

$$\mathcal{H} = \{h_\lambda \mid \lambda \in \Lambda\}$$

denote a set of hedges available to Skeptic in each round. For example in Section 4 we consider the set of symmetric call option type hedges ("strangle hedges", Chapter 10 of [3])

$$\mathcal{H} = \{(|x| - k)_+ \mid k = 0, 1, 2, \dots\},$$

where $x_+ = \max(0, x)$. $(|x| - k)_+$ is depicted in Figure 1. We assume that h_λ is available to Skeptic with a constant finite positive price v_{h_λ}. Skeptic is allowed to buy any amount of countable number of hedges $h_1, h_2, \dots$ from $\mathcal{H}$. If Skeptic buys $V_i \in \mathbb{R}$ units of h_i, $i = 1, 2, \dots$, then he is required that the sum $\sum_{i=1}^{\infty} V_i v_{h_i}$ converges to a finite value. Note that here for a set of hedges we are allowing Skeptic to sell a hedge ($V_i < 0$), whereas in the case of a single hedge Skeptic can obviously only buy the hedge. By allowing Skeptic to sell hedges, he can combine various hedges to construct a variety of hedges (Chapter 10 of [3], Section 9.3 of [1]). Based on these considerations we set up the following protocol.

The Unbounded Forecasting Game with a Set of Hedges

Protocol:

$\mathcal{K}_0 := 1$.
FOR $n = 1, 2, \dots$:
Skeptic announces $M_n \in \mathbb{R}$, $h_{n1}, h_{n2}, \dots \in \mathcal{H}$, $V_{n1}, V_{n2}, \dots \in \mathbb{R}$
s.t. $\sum_i V_{ni} v_{h_{ni}}$ converges to a finite value.
Reality announces $x_n \in \mathbb{R}$.
$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n + \sum_i V_{ni}(h_{ni}(x_n) - v_{h_{ni}})$.
END FOR

In our proofs we combine Skeptic's strategies to force intersection of events. From Section 3.2 of Shafer and Vovk (2001), a strategy $\mathcal{P}$ weakly forces an event E if it satisfies the collateral duty and $\limsup_n \sum_n \mathcal{K}_n^{\mathcal{P}}(\xi) = \infty$ for every $\xi \notin E$. In this case we also say that E happens almost surely. If $\limsup_n$ is replaced by $\lim_n$, then $\mathcal{P}$ forces E. Now consider two events E_1 and E_2. We say that a strategy $\mathcal{P}$ weakly forces E_2 *conditional on* E_1 if it satisfies the collateral duty and

$$\limsup_n \mathcal{K}_n^{\mathcal{Q}}(\xi) = \infty, \qquad \forall \xi \in E_1 \cap E_2^C.$$

Now we state the following lemma, which is slightly stronger than Lemma 3.2 of Shafer and Vovk (2001).

Lemma 2.1. *Suppose that Skeptic can weakly force E_1 and furthermore he can weakly force E_2 conditional on E_1. Then he can weakly force $E_1 \cap E_2$.*

Proof. Let $\mathcal{P}_1$ denote a strategy weakly forcing E_1 and let $\mathcal{P}_2$ denote a strategy weakly forcing E_2 conditional on E_1. Let $\mathcal{P} = (1/2)(\mathcal{P}_1 + \mathcal{P}_2)$. Note that

$$(E_1 \cap E_2)^C = E_1^C \cup (E_1 \cap E_2^C).$$

For $\xi \in E_1^C$ $\limsup_n \mathcal{K}_n^{\mathcal{P}}(\xi) = \infty$ since $\limsup_n \mathcal{K}_n^{\mathcal{P}_1}(\xi) = \infty$. Similarly for For $\xi \in E_1 \cap E_2^C$ $\limsup_n \mathcal{K}_n^{\mathcal{P}}(\xi) = \infty$ since $\limsup_n \mathcal{K}_n^{\mathcal{P}_2}(\xi) = \infty$. $\square$

It is clear that Lemma 2.1 can be generalized to the sequence of events $E_1, E_2, \ldots,$ such that E_i is weakly forced conditional on $E_1 \cap \cdots \cap E_{i-1}$.

Finally we state and discuss the game-theoretic martingale convergence theorem given in Lemma 4.5 of Shafer and Vovk (2001).

Lemma 2.2. *A non-negative martingale $\mathcal{K}^{\mathcal{P}}$ converges to a non-negative finite value almost surely.*

As seen from the proof of Lemma 4.5 of Shafer and Vovk (2001) this theorem is based on Doob's upcrossing lemma in the game-theoretic setting. We use this lemma in our proofs in an essential way. As discussed at the beginning of this section, when we say that $\mathcal{K}^{\mathcal{P}}$ is a non-negative martingale, it means that Skeptic observes his collateral duty (1) with the strategy $\mathcal{P}$ starting with a positive initial capital $\mathcal{K}_0 > 0$. In this case he can construct another strategy $\mathcal{Q}$ satisfying the collateral duty starting with an arbitrary small initial capital $\delta > 0$ such that

$$\lim_n \mathcal{K}_n^{\mathcal{Q}}(\xi) = \infty$$

whenever $\mathcal{K}_n^{\mathcal{P}}$ does not converge. As in Section 4.2 of Shafer and Vovk (2001) or Chapter 12 of Williams (1991) we use Lemma 2.2 in conjunction with Kronecker's lemma.

3 SLLN with a single hedge

In this section we give sufficient conditions for SLLN in the unbounded forecasting game with a single hedge. For simplicity we only consider symmetric hedge $h(x) = h(|x|)$ depending only on $|x|$. We assume several conditions for $h(|x|) \geq 0$.

$$\text{(A1)} \qquad \text{For some } c > 0, \qquad h(|x|) \geq |x| \text{ for } |x| \geq c. \tag{2}$$

$$\text{(A2)} \qquad \text{For some } c > 0 \text{ and for all } \alpha \geq 1$$

$$\frac{h(|x|)}{|x|^\alpha} \text{ is monotone increasing or decreasing for } |x| \geq c. \tag{3}$$

$$\text{(A3)} \qquad \text{For some } c > 0, \qquad \sum_{n>c}^{\infty} \frac{1}{h(n)} < \infty. \tag{4}$$

In our proof the condition (A3) is essential for SLLN with a single hedge, as shown in Proposition 3.1 below. On the other hand (A2) and the symmetry of h are assumed for convenience for our proofs. $c > 0$ in the conditions can be easily handled and for simplicity we assume $c = 0$ in our proofs below. By (A2), there exists some $\alpha_0 > 0$ such that $h(|x|)$ is monotone increasing in $|x|$ for $\alpha > \alpha_0$ and monotone decreasing in $|x|$ for $\alpha < \alpha_0$.

Now we state the following theorem.

Theorem 3.1. *Suppose that a single hedge $h(x)$ satisfying* (A1)–(A3) *is available to Skeptic. Then in the unbounded forecasting game with the single hedge $h(x)$, Skeptic can force $\bar{x}_n \to 0$.*

Take $h(x) = |x|^{1+\epsilon}$, $\epsilon > 0$, then (A1)–(A3) hold and SLLN is forced. SLLN is forced even for

$$h(x) = |x|(\log |x|)^2.$$

However as shown in Proposition 2.1, SLLN is not forced for $h(x) = |x|$.

Before starting the proof of Theorem 3.1 we show that the condition (A3) is also necessary for the existence of a strategy weakly forcing SLLN.

Proposition 3.1. *Consider $h(x) \geq 0$ with $h(0) = 0$ and $\sum_n 1/h(n) = \infty$. Then in the unbounded forecasting game with this single hedge $h(x)$, there exists no strategy $\mathcal{P}$ of Skeptic satisfying the collateral duty, such that $\lim_n \mathcal{K}_n^{\mathcal{P}} = \infty$ whenever $(x_1 + \cdots + x_n)/n \not\to 0$.*

Proof of this proposition is given in Appendix A.

The rest of this section is devoted to a proof of Theorem 3.1 in a series of lemmas. By Lemma 3.1 of Shafer and Vovk (2001) we only need to show that Skeptic can weakly force $\bar{x}_n \to 0$.

Lemma 3.1. *Let*

$$E_1 = \{\xi \mid \sum_n \frac{h(x_n)}{h(n)} < \infty\}.$$

Under the conditions (A1)–(A3) *Skeptic can force E_1.*

Proof. By (A3) let $C = \sum_n 1/h(n) < \infty$. Consider the following strategy $\mathcal{P}$

$$M_n \equiv 0, \qquad V_n = \frac{1}{C\nu h(n)}.$$

where $0 < \nu < \infty$ is the price of the hedge h. For this strategy, starting with the initial capital of $\mathcal{K}_0 = 1$, the capital process $\mathcal{K}_n$ is written as

$$\begin{aligned}
\mathcal{K}_n &= 1 + \sum_{i=1}^{n} \frac{1}{C\nu h(i)}(h(x_i) - \nu) \\
&= 1 - \frac{1}{C}\sum_{i=1}^{n}\frac{1}{h(i)} + \frac{1}{C\nu}\sum_{i=1}^{n}\frac{h(x_i)}{h(i)} \\
&\geq \frac{1}{C\nu}\sum_{i=1}^{n}\frac{h(x_i)}{h(i)}.
\end{aligned}$$

Therefore $\mathcal{P}$ satisfies the collateral duty and on E_1^C $\mathcal{K}_n$ diverges to $+\infty$. Therefore $\mathcal{P}$ forces E_1. $\square$

Note that the same argument with $C = \sum_n 1/n^2$ shows that Skeptic can force

$$E_1' = \{\xi \mid \sum_n \frac{h(x_n)}{n^2} < \infty\}. \tag{5}$$

Furthermore Lemma 3.1 implies the following Borel-Cantelli type result.

Lemma 3.2. *Let*

$$E_2 = \{\xi \mid |x_n| \geq n \text{ for only finite number of } n\}. \tag{6}$$

Under the conditions (A1)–(A3) *Skeptic can force* E_2*.*

Proof. By (A2) $h(|x|)/|x|$ is monotone. If it is monotone decreasing (A3) can not hold. Therefore $h(|x|)/|x|$ has to be monotone increasing and $h(|x|)$ is itself monotone increasing. Therefore for $z > 0$

$$\frac{h(z)}{h(n)} \geq I_{[n,\infty)}(z),$$

where $I_{[n,\infty)}(\cdot)$ is the indicator function of the interval $[n, \infty)$. It follows that $E_1 \subset E_2$. $\square$

It should be noted that this lemma is essentially the first part of Borel-Cantelli lemma. For convenience we state a game-theoretic version of the first part of Borel-Cantelli lemma. The proof is the same as in Lemma 3.1 and omitted.

Lemma 3.3. (The first part of Borel-Cantelli) *Let $E_1, E_2, \ldots$ be a sequence of events such that the sum of the upper probabilities is finite $\sum_n \bar{P}(E_n) < \infty$. Then Skeptic can force*

$$(\limsup_n E_n)^C = \{E_n \text{ only for finite } n\}.$$

The following lemma concerns the evaluation of the variance of truncated variables in the usual proof of SLLN.

Lemma 3.4. *Let*

$$E_3 = \{\xi \mid \sum_n \frac{x_n^2}{n^2} I_{\{|x_n| \le n\}} < \infty\}. \tag{7}$$

Under the conditions (A1)–(A3) *Skeptic can force* E_3*.*

Proof. First consider the case that $h(x)/x^2$ is monotone increasing. Then adjusting some constants we can assume $h(x) \ge x^2$ for all x without loss of generality. Then

$$\sum_n \frac{x_n^2}{n^2} I_{\{|x_n| \le n\}} \le \sum_n \frac{x_n^2}{n^2} \le \sum_n \frac{h(x_n)}{n^2}$$

and $E_1' \subset E_3$, where E_1' is given in (5). Therefore Skeptic can force E_3.

Next consider the case that $h(x)/x^2$ is monotone decreasing. For $0 < z \le n$ we have

$$\frac{h(z)}{z^2} \ge \frac{h(n)}{n^2}.$$

Multiplying both sides by $n^2/h(z)$ we have

$$\frac{z^2}{n^2} \le \frac{h(z)}{h(n)}.$$

Then

$$\sum_n \frac{x_n^2}{n^2} I_{\{|x_n| \le n\}} \le \sum_n \frac{h(x_n)}{h(n)}$$

and $E_1 \subset E_3$. □

From Lemma 3.2 and Lemma 3.4 Skeptic can force $E_2 \cap E_3$.

Lemma 3.5. *Let* $0 < c \le 1/[2(1 + \nu/h(1))]$*. Then for all* x

$$-c\frac{|x|}{n} + c\frac{h(x) - \nu}{h(n)} \ge -\frac{1}{2}.$$

Proof. Since $h(z)/z$ is increasing in $z > 0$, for $z \ge n$ we have $h(n)/n \le h(z)/z$. Multiplying by $z/h(n)$ we have

$$\frac{h(z)}{h(n)} - \frac{z}{n} \ge 0, \quad z \ge n.$$

For $0 \le z \le n$ obviously

$$\frac{h(z)}{h(n)} - \frac{z}{n} \ge -1.$$

Therefore for all $z \ge 0$ we have

$$\frac{h(z) - \nu}{h(n)} - \frac{z}{n} \ge -1 - \frac{\nu}{h(n)} \ge -1 - \frac{\nu}{h(1)}$$

and this proves the lemma. □

Finally the following lemma proves Theorem 3.1 by Kronecker's lemma.

Lemma 3.6. *Let*

$$E_4 = \{\xi \mid \sum_n \frac{x_n}{n} \text{ converges to a finite value}\}. \tag{8}$$

Under the conditions (A1)–(A3) *Skeptic can weakly force E_4 conditional on E_1.*

Proof. Let $0 < \epsilon \le 1/[2(1+\nu/h(1))]$. Consider the following strategy $\mathcal{P}^+$:

$$M_n = \epsilon \mathcal{K}_{n-1}\frac{1}{n}, \quad V_n = \epsilon \mathcal{K}_{n-1}\frac{1}{h(n)}.$$

Then by Lemma 3.5

$$\mathcal{K}_n = \mathcal{K}_{n-1}(1+\epsilon\frac{x_n}{n} + \epsilon\frac{h(x_n)-\nu}{h(n)}) \ge \frac{1}{2}\mathcal{K}_{n-1}$$

and $\mathcal{P}^+$ satisfies the collateral duty. Similarly the strategy $\mathcal{P}^-$ with $M_n = -\epsilon\mathcal{K}_{n-1}/n, V_n = \epsilon\mathcal{K}_{n-1}/h(n)$ satisfies the collateral duty. By the game-theoretic martingale convergence theorem (Lemma 2.2) both $\mathcal{K}_n^{\mathcal{P}^+}$ and $\mathcal{K}_n^{\mathcal{P}^-}$ converge to a non-negative finite value almost surely. Then both $\log \mathcal{K}_n^{\mathcal{P}^+}$ and $\log \mathcal{K}_n^{\mathcal{P}^-}$ converge to a finite value or $-\infty$ almost surely.

As in Lemma 3.3 of Shafer and Vovk (2001) we use the inequality $t \ge \log(1+t) \ge t - t^2$ for all $t \ge -1/2$. Then the logarithm of the capital process for $\mathcal{P}^+$ starting with $\mathcal{K}_0 = 1$ is bounded as

$$\begin{aligned}\epsilon\sum_{i=1}^n (\frac{x_i}{i} - \frac{h(x_i)-\nu}{h(i)}) &\ge \log \mathcal{K}_n^{\mathcal{P}^+} \\ &\ge \epsilon\sum_{i=1}^n (\frac{x_i}{i} - \frac{h(x_i)-\nu}{h(i)}) - \epsilon^2 \sum_{i=1}^n (\frac{x_i}{i} - \frac{h(x_i)-\nu}{h(i)})^2. \end{aligned} \tag{9}$$

On E_1, each of the following infinite sums is finite.

$$\sum_n \frac{h(x_n)}{h(n)}, \quad \sum_n \frac{\nu}{h(n)}, \quad \sum_n \frac{x_n^2}{n^2}, \quad \sum_n \frac{h(x_n)^2}{h(n)^2}, \quad \sum_n \frac{\nu^2}{h(n)^2}.$$

By the inequality

$$(a_1 + \cdots + a_m)^2 \le m(a_1^2 + \cdots + a_m^2)$$

on E_1 the second term on the right-hand side of (9) converges to a finite value:

$$\sum_{n=1}^\infty (\frac{x_n}{n} - \frac{h(x_n)-\nu}{h(n)})^2 < \infty.$$

Therefore conditional on E_1 $\mathcal{P}^+$ weakly forces

$$\limsup_n \sum_{i=1}^n \frac{x_i}{i} < \infty.$$

Similarly conditional on E_1 $\mathcal{P}^-$ weakly forces

$$\liminf_n \sum_{i=1}^n \frac{x_i}{i} < -\infty.$$

It follows that the case $\lim \log \mathcal{K}_n^{\mathcal{P}^+} = -\infty$ is eliminated and $\log \mathcal{K}_n^{\mathcal{P}^+}$ converges to a finite value almost surely.

Now consider (9) for the interval $n \le i \le n'$. Then

$$\epsilon \sum_{i=n}^{n'} \big(\frac{x_i}{i} - \frac{h(x_i)-\nu}{h(i)}\big) \ge \log \mathcal{K}_{n'}^{\mathcal{P}^+} - \log \mathcal{K}_{n-1}^{\mathcal{P}^+}$$
$$\ge \epsilon \sum_{i=n}^{n'} \big(\frac{x_i}{i} - \frac{h(x_i)-\nu}{h(i)}\big) - \epsilon^2 \sum_{i=n}^{n'} \big(\frac{x_i}{i} - \frac{h(x_i)-\nu}{h(i)}\big)^2.$$

Now by Cauchy criterion we see that $\sum_n x_n/n$ converges almost surely. □

4 SLLN with countable hedges

In this section we prove a version of game-theoretic SLLN which corresponds to the usual measure-theoretic SLLN for the sample average of i.i.d. random variables with finite expectation. As shown in Proposition 2.1, the availability of a single $h(x) = |x|$ is not sufficient. It seems that an essential ingredient of measure-theoretic proofs of SLLN for this case is that the expected values of truncation are uniformly bounded by the assumption of identical distribution. Hence we consider that countable number of hedges are available with constant prices at each round of the game. We assume that the prices are given in such a way that the game is coherent, i.e. the game does not present an arbitrage opportunity to Skeptic (see Section 7.1 of [7] or [8]).

As mentioned in Section 2, for our game-theoretic version of SLLN we assume that the set of symmetric call option type hedges with integral exercise prices $k = 0, 1, 2, \ldots$

$$\mathcal{H} = \{h_k(x) = (|x| - k)_+ \mid k = 0, 1, 2, \ldots\} \tag{10}$$

are available to Skeptic. In particular $|x| = (|x| - 0)_+$ is available to Skeptic. Let ν_k denote the price of $h_k(x) = (|x| - k)_+$. We also assume that Skeptic is allowed to sell hedges and combine them, as long as he observes his collateral duty. For example he can create a new hedge

$$(|x| - k)_+ - (|x| - k - 1)_+ = \begin{cases} 0, & |x| \le k \\ |x| - k, & k < |x| \le k + 1 \\ 1, & |x| > k + 1. \end{cases}$$

This new hedge carries the price of $\nu_k - \nu_{k+1} \ge 0$. We may call this hedge "symmetric bull spread" (c.f. Chapter 10 of [3]).

For truncation arguments below we also consider "symmetric trapezoidal hedge". For $k \geq 1$ define

$$T_k(x) = (|x| - (k-1))_+ - (|x| - k)_+ - \big((|x| - (k+1))_+ - (|x| - (k+2))_+\big)$$
$$= \begin{cases} 0, & |x| \leq k-1 \\ |x| - (k-1), & k-1 < |x| \leq k \\ 1, & k < |x| \leq k+1 \\ k+2-|x|, & k+1 < |x| \leq k+2 \\ 0, & k+2 < |x| \end{cases}$$
$$\geq I_{[k,k+1]}(|x|)$$

with the price $\mu_k = \nu_{k+2} - \nu_{k+1} - \nu_k + \nu_{k-1}$. For $k = 0$, $T_0(x) = 1 - ((|x|-1)_+ - (|x|-2)_+)$, which is a single trapezoid. Symmetric bull spread and the positive side of symmetric trapezoidal hedge are depicted in Figure 2 and Figure 3, respectively.

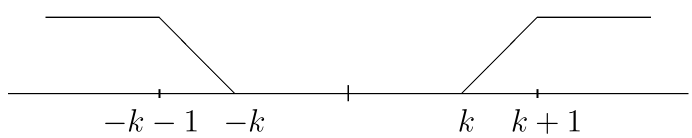

Figure 2: Symmetric bull spread

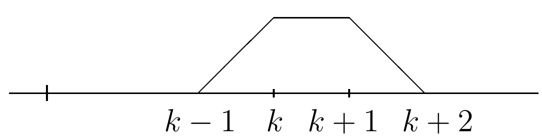

Figure 3: Symmetric trapezoidal hedge (positive side only)

Now we state the following theorem.

Theorem 4.1. *Suppose that the set of hedges $\mathcal{H} = \{h_k(x) = (|x| - k)_+ \mid k = 0, 1, 2, \dots\}$ are available to Skeptic. Then in the unbounded forecasting game with $\mathcal{H}$ Skeptic can force $\bar{x}_n \to 0$.*

The rest of this section is devoted to a proof of this theorem. As in the previous section we prove it by a series of lemmas.

Lemma 4.1. *Under the condition of Theorem 4.1 Skeptic can force E_2 in* (6).

Proof. For $k \geq 1$, $(|x| - k + 1)_+ - (|x| - k)_+ \geq I_{[k,\infty)}(|x|)$ and

$$\sum_{n=1}^{\infty} \big((|x_n| - n + 1)_+ - (|x_n| - n)_+\big) \geq \sum_{n=1}^{\infty} I_{[n,\infty)}(|x_n|).$$

The left-hand side can be bought with the total finite price of ν_0. The rest of the proof is the same as in Lemma 3.1. $\square$

Lemma 4.2. *Under the condition of Theorem 4.1 Skeptic can force* E_3 *in* (7).

Proof. At round n Skeptic is to buy $(k+1)^2$ units of the symmetric trapezoidal hedge T_k for each $k = 0, 1, \ldots, n-1$. We note

$$\sum_{k=0}^{n-1}(k+1)^2 T_k(x_n) \ge x_n^2 I_{\{|x_n|\le n\}}.$$

Dividing the above by n^2 and summing up over all rounds $n = 1, 2, \ldots,$ we have

$$\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{(k+1)^2}{n^2}T_k(x_n) \ge \sum_{n=1}^{\infty}\frac{x_n^2}{n^2}I_{\{|x_n|\le n\}}.$$

Now we evaluate the total price of the left-hand side. Since T_k is available at each round, the price is the same if we replace x_n by x_1 in T_k. Then

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{(k+1)^2}{n^2}T_k(x_1) &= \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{l=1}^{n} l^2 T_{l-1}(x_1) = \sum_{l=1}^{\infty} l^2 T_{l-1}(x_1)\sum_{n=l}^{\infty}\frac{1}{n^2}\\ &\le 2\sum_{l=1}^{\infty} l T_{l-1}(x_1) \le 6|x_1|.\end{aligned}$$

As noted above $|x_1|$ is available to Skeptic with finite price ν_0, so that the left-hand side is also available to him with the total finite price

$$\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\frac{(k+1)^2}{n^2}\mu_k \le 6\nu_0.$$

The rest of the proof is the same as in Lemma 3.1. $\square$

In the following x_n hedged by $(|x_n| - n)_+$ is denoted as

$$x_{n,n} = x_n + (|x_n| - n)_+ = \begin{cases} -n, & x_n < -n,\\ x_n, & -n \le x_n \le n,\\ 2x_n - n, & x_n > n.\end{cases}$$

This has the price ν_n. Similarly we denote $\tilde{x}_{n,n} = -x_n + (|x_n| - n)_+$ which is $-x_n$ hedged by $(|x_n| - n)_+$. Note that

$$x_{n,n} \ge -n, \qquad \tilde{x}_{n,n} \ge -n.$$

On E_2, $x_{n,n}$ and $x_n I_{\{|x_n|\le n\}}$ differ only for finite number of n. Therefore conditional on E_2, Skeptic can force

$$E_3' = \{\xi \mid \sum_n \frac{x_{n,n}^2}{n^2} < \infty\}, \qquad E_3'' = \{\xi \mid \sum_n \frac{\tilde{x}_{n,n}^2}{n^2} < \infty\}. \tag{11}$$

Lemma 4.3. *Under the condition of Theorem 4.1 and conditional on E_2, Skeptic can force*

$$E_5 = \{\xi \mid \sum_n \frac{(x_{n,n} - \nu_n)^2}{n^2} < \infty\}$$

Proof. Since $(x_{n,n} - \nu_n)^2 \le 2x_{n,n}^2 + 2\nu_n^2$

$$\sum_{n=1}^N \frac{(x_{n,n} - \nu_n)^2}{n^2} \le 2\sum_{n=1}^N \frac{x_{n,n}^2}{n^2} + 2\sum_{n=1}^N \frac{\nu_n^2}{n^2} \le 2\sum_{n=1}^N \frac{x_{n,n}^2}{n^2} + 2\nu_0^2 \frac{\pi^2}{6}$$

By (11), conditional on E_2, Skeptic can force $\sum_{n=1}^\infty x_{n,n}^2/n^2 < \infty$. Therefore conditional on E_2, he can force E_5 □

Similarly Skeptic can force E_5 with $x_{n,n}$ replaced by $\tilde{x}_{n,n}$.

Finally the following lemma proves Theorem 4.1 in conjunction with Kronecker's lemma.

Lemma 4.4. *Under the condition of Theorem 4.1 Skeptic can weakly force*

$$E_4' = \{\xi \mid \sum_n \frac{x_{n,n} - \nu_n}{n} \text{ converges to a finite value}\}$$

conditional on $E_2 \cap E_3'$.

Proof. We take ϵ as

$$0 < \epsilon < \frac{1}{2(1+\nu_0)},$$

and consider Skeptic's strategy betting $\epsilon \mathcal{K}_{n-1}/n$ on $x_{n,n} - \nu_n$ at round n. Then his capital at the end of round n is

$$\mathcal{K}_n = \mathcal{K}_{n-1}(1 + \frac{\epsilon}{n}(x_{n,n} - \nu_n)) = \mathcal{K}_0 \prod_{i=1}^n (1 + \frac{\epsilon}{i}(x_{i,i} - \nu_i)).$$

By the choice of ϵ and $|x_{i,i}/i| \le 1$,

$$\frac{\epsilon}{i}(x_{i,i} - \nu_i) \ge -\frac{1}{2},$$

so that from $\log(1+t) \ge t - t^2$ for $t \ge -1/2$, his log capital is bounded from below as

$$\log \mathcal{K}_n \ge \log \mathcal{K}_0 + \epsilon \sum_{i=1}^n \frac{x_{i,i} - \nu_i}{i} - \epsilon^2 \sum_{i=1}^n \frac{(x_{i,i} - \nu_i)^2}{i^2}.$$

In the right-hand side the third term is bounded on E_5. By considering this inequality for the interval $n \le i \le n'$, we have

$$\log \mathcal{K}_{n'} - \log \mathcal{K}_{n-1} \ge \epsilon \sum_{i=n}^{n'} \frac{x_{i,i} - \nu_i}{i} - \epsilon^2 \sum_{i=n}^{n'} \frac{(x_{i,i} - \nu_i)^2}{i^2}.$$

As in the proof of Lemma 3.6, considering both $x_{n,n}$ and $\tilde{x}_{n,n}$, $\log \mathcal{K}_n$ converges to a finite limit almost surely, and thus by Cauchy criterion we see that

$$\sum_{i=1}^{n} \frac{x_{i,i} - \nu_i}{i}$$

converges almost surely. □

As proved in Appendix B, $\nu_n \to 0$ as $n \to \infty$. Then by Kronecker's lemma we have

$$\frac{1}{n}\sum_{i=1}^{n}(x_{i,i} - \nu_i) = \frac{1}{n}\sum_{i=1}^{n} x_{i,i} - \frac{1}{n}\sum_{i=1}^{n} \nu_i \to 0 \quad \text{as } n \to \infty.$$

In the above, $(1/n)\sum_{i=1}^{n} \nu_i \to 0$ so that $\sum_{i=1}^{n} x_{i,i}/n$ also converges to 0. Since x_n and $x_{n,n}$ differ only for finite number of n on E_2, it is concluded that $\bar{x}_n$ converges to 0 almost surely. This completes the proof of Theorem 4.1.

5 Marcinkiewicz-Zygmund strong law

In this section we consider a remarkable generalization by Marcinkiewicz and Zygmund (See [2], [6]) of Kolmogorov's measure-theoretic SLLN for i.i.d. random variables with finite expected value $E|x_n| < \infty$. Marcinkiewicz-Zygmund strong law asserts that for i.i.d. random variables $x_1, x_2, \dots$ with $E|x_n|^r < \infty$ for $0 < r < 2$ and $Ex_n = 0$ when $1 \le r < 2$, the following measure-theoretic SLLN holds

$$\frac{x_1 + \cdots + x_n}{n^{1/r}} \to 0 \quad \text{as } n \to \infty \quad a.s.$$

Considering the meaning of the hedge $|x|^r$ for x in betting games, we treat the case $1 < r < 2$ and for this case establish a game-theoretic version of Marcinkiewicz-Zygmund SLLN. As noted in Proposition 2.1, the availability of a single $h(x) = |x|^r$ is again not sufficient. Hence here, for the game-theoretic Marcinkiewicz-Zygmund SLLN we assume that the following set of hedges are available to Skeptic.

$$\mathcal{H}_r = \mathcal{H}_{1r} \cup \mathcal{H}_{2r}, \quad \text{where} \quad \begin{cases} \mathcal{H}_{1r} = \{h_{kr}(x) = (|x|^r - k)_+ \mid k = 0, 1, 2, \dots\} \\ \mathcal{H}_{2r} = \{h_{k^{1/r}}(x) = (|x| - k^{1/r})_+ \mid k = 0, 1, 2, \dots\} \end{cases} . \tag{12}$$

Let ν_{kr} denote the price of $h_{kr}(x) = (|x|^r - k)_+$ and let $\nu_{k^{1/r}}$ denote the price of $h_{k^{1/r}}(x) = (|x| - k^{1/r})_+$. Also assuming that Skeptic is allowed to sell and combine these hedges within his collateral duty, we state the following theorem.

Theorem 5.1. *Let $1 < r < 2$. Suppose that the set of hedges $\mathcal{H}_r$ in (12) is available to Skeptic. Then in the unbounded forecasting game with $\mathcal{H}_r$ Skeptic can force $(x_1 + \cdots + x_n)/n^{1/r} \to 0$.*

Remark 5.1. *In this theorem* $\mathcal{H}_r$ *consists of two sets of hedges* $\mathcal{H}_{1r}$ *and* $\mathcal{H}_{2r}$*.* $\mathcal{H}_{2r}$ *is included in* $\mathcal{H}$ *just for convenience. Each* $h_{k^{1/r}}(x)$ *can be superreplicated and underreplicated by an infinite combination of hedges from* $\mathcal{H}_{1r}$ *and the theorem holds without* $\mathcal{H}_{2r}$*. Since this makes the proof considerably messier, we include* $\mathcal{H}_{2r}$ *in the set of hedges. We give more discussion on this point in Section 6.*

The proof of Theorem 5.1 proceeds almost in the same way as that of Theorem 4.1. However we have to make different uses of hedges from $\mathcal{H}_{1r}$ and from $\mathcal{H}_{2r}$. At first we enumerate relevant events.

$$E_{2r} = \{\xi \mid |x_n|^r \geq n \text{ for only finite number of } n\}.$$

$$E_{3r} = \{\xi \mid \sum_n \frac{x_n^2}{n^{2/r}} I_{\{|x_n|^r \leq n\}} < \infty\}.$$

Lemma 5.1. *Under the condition of Theorem 5.1 Skeptic can force* E_{2r}*.*

Proof. For $k \geq 1$, $(|x|^r - (k-1))_+ - (|x|^r - k)_+ \geq I_{[k,\infty)}(|x|^r)$ and

$$\sum_{n=1}^{\infty} \big((|x_n|^r - (n-1))_+ - (|x_n|^r - n)_+\big) \geq \sum_{n=1}^{\infty} I_{[n,\infty)}(|x_n|^r).$$

The left-hand side can be bought with the total finite price of ν_{0r}. The rest of the proof is the same as in Lemma 3.1. □

Lemma 5.2. *Under the condition of Theorem 5.1 Skeptic can force* E_{3r}*.*

Proof. Consider the following trapezoidal hedge

$$\begin{aligned} T_{kr}(x) &= (|x|^r - (k-1))_+ - (|x|^r - k)_+ - \big((|x|^r - (k+1))_+ - (|x|^r - (k+2))_+\big) \\ &= \begin{cases} 0, & |x|^r \leq k-1 \\ |x|^r - (k-1), & k-1 < |x|^r \leq k \\ 1, & k < |x|^r \leq k+1 \\ k+2-|x|^r, & k+1 < |x|^r \leq k+2 \\ 0, & k+2 < |x|^r \end{cases} \\ &\geq I_{[k,k+1]}(|x|^r) \end{aligned}$$

with the price $\mu_{kr} = \nu_{k+2,r} - \nu_{k+1,r} - \nu_{kr} + \nu_{k-1,r}$.

At round n Skeptic is to buy $(k+1)^2$ units of the hedge T_{kr} for each $k = 0, 1, \ldots, n-1$. We note

$$\sum_{k=0}^{n-1} (k+1)^2 T_{kr}(x_n) \geq x_n^2 I_{\{|x_n|^r \leq n\}}.$$

Dividing the above by $n^{2/r}$ and summing up over all rounds $n = 1, 2, \dots$, we have

$$\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(k+1)^2}{n^{2/r}} T_{kr}(x_n) \geq \sum_{n=1}^{\infty} \frac{x_n^2}{n^{2/r}} I_{\{|x_n|^r \leq n\}}.$$

As in the previous section, for the consideration of the total price, we can replace $T_{kr}(x_n)$ by $T_{kr}(x_1)$. Then the left-hand side can be evaluated as

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(k+1)^2}{n^{2/r}} T_{kr}(x_1) &= \sum_{n=1}^{\infty} \frac{1}{n^{2/r}} \sum_{l=1}^{n} l^2 T_{l-1,r}(x_1) = \sum_{l=1}^{\infty} l^2 T_{l-1,r}(x_1) \sum_{n=l}^{\infty} \frac{1}{n^{2/r}} \\
&\leq \frac{2^{(2/r)-1}}{(2/r)-1} \sum_{l=1}^{\infty} \frac{1}{l^{(2/r)-1}} l^2 T_{l-1,r}(x_1) \\
&\leq \frac{2^{(2/r)-1}}{(2/r)-1} \sum_{l=1}^{\infty} \frac{1}{l^{(2/r)-1}} (l^{1/r})^{2-r} l^r T_{l-1,r}(x_1) \\
&\leq \frac{2^{(2/r)-1}}{(2/r)-1} \sum_{l=1}^{\infty} l^r T_{l-1,r}(x_1) \leq \frac{3 \cdot 2^{(2/r)-1}}{(2/r)-1} |x_1|^r.
\end{aligned}$$

Since $|x_1|^r$ is available to Skeptic with finite price ν_{0r}, the left-hand side is also available to him with the total finite price

$$\sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(k+1)^2}{n^{2/r}} \mu_{kr} \leq \frac{3 \cdot 2^{(2/r)-1}}{(2/r)-1} \nu_{0r}.$$

□

So far we have used hedges from $\mathcal{H}_{1r}$ for forcing various events. In the following x_n will be hedged by elements from $\mathcal{H}_{2r}$. We hedge x_n by $h_{n^{1/r}}(x_n) = (|x_n| - n^{1/r})_+$. Write

$$x_{nn,r} = x_n + (|x_n| - n^{1/r})_+.$$

This has the price $\nu_{n^{1/r}}$. On E_{2r}, $x_{nn,r}$ and $x_n I_{\{|x_n|^r \leq n\}}$ differ only for finite number of n. Therefore conditional on E_{2r}, Skeptic can force

$$E'_{3r} = \{\xi \mid \sum_n \frac{x_{nn,r}^2}{n^{2/r}} < \infty\}. \tag{13}$$

Lemma 5.3. *Under the condition of Theorem 5.1 and conditional on E_{2r}, Skeptic can force*

$$E_{5r} = \{\xi \mid \sum_n \frac{(x_{nn,r} - \nu_{n^{1/r}})^2}{n^{2/r}} < \infty\}.$$

Proof.

$$\sum_n \frac{(x_{nn,r} - \nu_{n^{1/r}})^2}{n^{2/r}} \le 2\sum_n \frac{x_{nn,r}^2}{n^{2/r}} + 2\sum_n \frac{\nu_{n^{1/r}}^2}{n^{2/r}}.$$

Both terms are finite on E_{2r}. □

Now we use the ϵ-strategy as before.

Lemma 5.4. *Under the condition of Theorem 5.1 Skeptic can weakly force*

$$E'_{4r} = \{\xi \mid \sum_n \frac{x_{nn,r} - \nu_{n^{1/r}}}{n^{1/r}} \text{ converges to a finite value}\}$$

conditional on $E_{2r} \cap E'_{3r}$.

Proof. We take ϵ as

$$0 < \epsilon < \frac{1}{2(1+\nu_0)},$$

and consider Skeptic's strategy betting $\epsilon\mathcal{K}_{n-1}/n$ on $x_{n,n} - \nu_{n^{1/r}}$ at round n. Then his capital at the end of round n is

$$\mathcal{K}_n = \mathcal{K}_{n-1}(1 + \frac{\epsilon}{n}(x_{nn,r} - \nu_{n^{1/r}})) = \mathcal{K}_0 \prod_{i=1}^n (1 + \frac{\epsilon}{i}(x_{ii,r} - \nu_{i^{1/r}})).$$

By the choice of ϵ and $|x_{ii,r}/i^{1/r}| \le 1$,

$$\frac{\epsilon}{i^{1/r}}(x_{ii,r} - \nu_{i^{1/r}}) \ge -\frac{1}{2},$$

so that from $\log(1+t) \ge t - t^2$ for $t \ge -1/2$, his log capital is bounded from below as

$$\log \mathcal{K}_n \ge \log \mathcal{K}_0 + \epsilon \sum_{i=1}^n \frac{x_{ii,r} - \nu_{i^{1/r}}}{i^{1/r}} - \epsilon^2 \sum_{i=1}^n \frac{(x_{ii,r} - \nu_{i^{1/r}})^2}{i^{2/r}}.$$

In the right-hand side the third term is bounded on E_{5r}. By considering this inequality for the interval $n \le i \le n'$,

$$\log \mathcal{K}_{n'} - \log \mathcal{K}_{n-1} \ge \epsilon \sum_{i=n}^{n'} \frac{x_{ii,r} - \nu_{i^{1/r}}}{i^{1/r}} - \epsilon^2 \sum_{i=n}^{n'} \frac{(x_{ii,r} - \nu_{i^{1/r}})^2}{i^{2/r}}.$$

In the above $\log \mathcal{K}_n$ converges to a finite limit almost surely, and thus as before

$$\sum_{i=1}^n \frac{x_{ii,r} - \nu_{i^{1/r}}}{i^{1/r}}$$

converges almost surely. □

We now need to take care of $n^{-1/r}\sum_{i=1}^n \nu_{i^{1/r}}$.

Lemma 5.5.
$$\frac{1}{n^{1/r}}\sum_{i=1}^n \nu_{i^{1/r}} \to 0 \qquad \text{as} \quad n\to\infty.$$

Proof. Since $r>1$
$$\big(1-\frac{n^{1/r}}{|x|}\big)_+ \le \big(1-\frac{n}{|x|^r}\big)_+, \qquad \forall x.$$
Also for $|x|\ge n^{1/r}$ we have $|x|^{r-1}\ge n^{1-1/r}$. Therefore
$$\begin{aligned}(|x|-n^{1/r})_+ &= |x|\big(1-\frac{n^{1/r}}{|x|}\big)_+\\ &\le \frac{|x|^{r-1}}{n^{1-1/r}}|x|\big(1-\frac{n}{|x|^r}\big)_+\\ &= n^{1/r-1}(|x|^r-n)_+\end{aligned}$$
It follows that the prices of $(|x|-n^{1/r})_+$ and $(|x|^r-n)_+$ have to satisfy
$$\nu_{n^{1/r}} \le n^{1/r-1}\nu_{nr}.$$
Therefore
$$\frac{1}{n^{1/r}}\sum_{i=1}^n \nu_{i^{1/r}} \le \frac{1}{n^{1/r}}\sum_{i=1}^n i^{1/r-1}\nu_{ir}.$$
Note that $\nu_{ir}\to 0$ as $i\to\infty$ by the argument in Appendix B. Then the right-hand side converges to 0 as $n\to\infty$ by Cesàro's lemma (12.6 of [10]). $\square$

Now by an extended form of Kronecker's lemma (12.7 of [10])
$$\frac{1}{n^{1/r}}\sum_{i=1}^n (x_{ii,r}-\nu_{i^{1/r}}) = \frac{1}{n^{1/r}}\sum_{i=1}^n x_{ii,r} - \frac{1}{n^{1/r}}\sum_{i=1}^n \nu_{i^{1/r}} \to 0 \quad \text{as } n\to\infty.$$
so that $\sum_{i=1}^n x_{i,i}/n^{1/r}$ also converges to 0. Since x_n and $x_{nn,r}$ differ only for finite number of n on E_{2r}, it follows that $(x_1+\cdots+x_n)/n^{1/r}$ converges to 0 almost surely. This completes the proof of Theorem 5.1.

6 Some discussions

In this paper we proved various game-theoretic versions of SLLN for unbounded variables. In Section 4 we proved a version corresponding to the sample average of i.i.d. measure-theoretic random variables. There we assumed availability of countable symmetric call option type hedges. We chose this set of hedges for convenience and concreteness. Other

choices are equally conceivable, as long as the set of hedges is rich enough to produce step-function type hedges (cf. Figure 2).

We might as well assume that if a hedge h is available to Skeptic, all other hedges weaker then h are available to him with price no more than that of h. We call a set of hedges $\mathcal{H}$ weakly closed if

$$h \in \mathcal{H},\ 0 \le g(x) \le h(x), \forall x \in \mathbb{R} \quad \Rightarrow \quad g \in \mathcal{H}.$$

We might argue that this is a reasonable assumption, because if h is available to Skeptic, he can ask to buy a weaker g with the same price as h and someone should be willing to sell g to Skeptic with the same price, because it presents an arbitrage opportunity to the seller. If $\mathcal{H}$ is weakly closed, then for each $t \in \mathbb{R}$

$$I_{(-\infty,t]}(x)$$

has to be available to Skeptic. This shows that if $\mathcal{H}$ is weakly closed, then the entire distribution function of the Reality's move x is priced in the game. The assumption of weakly closed $\mathcal{H}$ seems to be too strong from game-theoretic viewpoints. However we should mention that in measure-theoretic proofs the probability distribution is assumed and truncation is freely used.

The discussion on generality of probability games in Chapter 8 of Shafer and Vovk (2001) convincingly argues that measure-theoretic martingales can be reduced to game-theoretic martingales. If we interpret Theorem 3.1 in measure-theoretic terms and just rewrite our proof in measure-theoretic terms, we obtain the following result.

Proposition 6.1. *Let $\{Y_n\}$ be a measure-theoretic martingale adapted to an increasing family of σ-fields $\{\mathcal{F}_n\}$. Let h be a function satisfying* (A1)–(A3)*. If the measure-theoretic conditional expectation*

$$E[h(Y_n - Y_{n-1}) \mid \mathcal{F}_{n-1}]$$

is uniformly bounded, then $P(\lim_n Y_n/n = 0) = 1$.

Except for Proposition 2.1 we could avoid measure theory to establish our theorems. We believe that this again shows effectiveness of game-theoretic proofs as we have shown in our previous works ([5], [4]).

For the Marcinkiewicz-Zygmund strong law in Section 5 we have given a game-theoretic proof for $r > 1$. We also assumed availability of two kinds of hedges for convenience as we discussed in Remark 5.1. If we make the blanket assumption that $\mathcal{H}$ is weakly closed, then we believe that measure-theoretic proof of the Marcinkiewicz-Zygmund strong for $0 < r < 1$ can be translated to game-theoretic proof without too many modifications. From game-theoretic viewpoint however, the case $r < 1$ does not seem to be natural.

A Proofs of Proposition 2.1 and Proposition 3.1

Proof of Proposition 2.1. We argue by contradiction. Suppose there exists Skeptic's strategy $\mathcal{P}$ which allows Skeptic to observe his collateral duty with the initial capital

$\mathcal{K}_0 = 1$ and $\lim_n \mathcal{K}_n^{\mathcal{P}} = \infty$ whenever $s_n/n^{1/r} \not\to 0$, where $s_n = x_1 + \cdots + x_n$. Consider a random strategy of Reality, where each x_n, $n > \nu$, is independently chosen as

$$P(x_n = 0) = 1 - \frac{\nu}{n},\ P(x_n = n^{1/r}) = P(x_n = -n^{1/r}) = \frac{\nu}{2n}.$$

Here ν is the price of $h(x) = |x|^r$. Then by the second part of measure-theoretic Borel-Cantelli lemma

$$1 = P(|x_n| = n^{1/r} \ \ i.o.) = P(|x_n|/n^{1/r} = 1 \ \ i.o.). \tag{14}$$

Note that if $s_n/n^{1/r} \to 0$, then $x_n/n^{1/r} \to 0$ because

$$\frac{s_n}{n^{1/r}} = \Big(\frac{n-1}{n}\Big)^{1/r} \frac{s_{n-1}}{(n-1)^{1/r}} + \frac{x_n}{n^{1/r}}.$$

Therefore (14) implies that $P(s_n/n^{1/r} \to 0) = 0$. Then by our assumption $P(\mathcal{K}_n^{\mathcal{P}} \to \infty) = 1$. However under the randomized strategy of Reality $\mathcal{K}_n^{\mathcal{P}}$ is a measure-theoretic non-negative martingale and its measure-theoretic expectation is $E(\mathcal{K}_n^{\mathcal{P}}) = \mathcal{K}_0 = 1$. Then by Doob's martingale inequality (e.g. Theorem 14.6 of [10])

$$P(\max_{k\le n} \mathcal{K}_k^{\mathcal{P}} \ge c) \le \frac{1}{c}, \quad \forall c > 0, \forall n.$$

and $P(\sup_n \mathcal{K}_n^{\mathcal{P}} \ge c) \le 1/c$. But this contradicts $P(\mathcal{K}_n^{\mathcal{P}} \to \infty) = 1$. □.

Proof of Proposition 3.1. Consider a random strategy of Reality, where each x_n for n, $h(n) > \nu$, is independently chosen as

$$P(x_n = 0) = 1 - \frac{\nu}{h(n)},\ P(x_n = n) = P(x_n = -n) = \frac{\nu}{2h(n)}.$$

The rest of the proof is the same as the proof of Proposition 2.1. □

B Proof of the fact $\lim_{k\to\infty} \nu_k = 0$

Consider the identity for $x \in \mathbb{R}$:

$$|x| = \sum_{k=0}^{\infty} \Big((|x| - k)_+ - (|x| - k - 1)_+\Big). \tag{15}$$

For each real x, the right-hand side is actually a finite sum and there is no question on the convergence. On the other hand consider the identity

$$\nu_0 = \sum_{k=0}^{K-1} (\nu_k - \nu_{k+1}) + \nu_K.$$

Since $\{\nu_k\}$ is a monotone non-increasing sequence of non-negative reals

$$c = \lim_{K \to \infty} \nu_K \geq 0$$

exists. If $c > 0$ then, $\nu_0 > \sum_{k=0}^{\infty}(\nu_k - \nu_{k+1})$. But then Skeptic can sell $|x|$ and buy the right-hand side of (15) and he is certain to make money. This contradicts the assumption of coherence.

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